Concluding remarks on general position.

Def 1.13

For K a finite complex of dimension $k < n \leq 3$ and M a PL manifold, a map

$f : \|K\| \to \operatorname{Int} M$

is in general position w.r.t. K iff $\exists$ a finite collection $B_1, \ldots, B_p$ of PL n-cells in M with $f(|K|) \subseteq \bigcup \operatorname{Int} B_j$

and affine embeddings

$h_i : B_i \longrightarrow \mathbb{R}^n$ with $f^{-1}(B_i)$

a subcomplex of K and

$h_i f|_{f^{-1}(B_i)} : f^{-1}(B_i) \longrightarrow \mathbb{R}^n$

a general position maps for

$i \in \{1, \dots, p\}$.

<u>Theorem 1.4</u>

Suppose K is a finite complex

of dimension $k < n \leq 3$,

A, B, C are subcomplexes of K

with $A \cup B \cup C = K$ and $A \cap C = \emptyset$.

Then given an n-mfld M,

a PL map $g : |K| \longrightarrow M$ with

$g|_{|B|}$ in general position w.r.t.

B (assuming $g(|B|) \subseteq \text{Int} M$),

and $\varepsilon > 0$, $\exists$ a PL map

$f : |K| \longrightarrow M$ satisfying:

a) $d(f(x), g(x)) < \varepsilon \quad \forall x \in |K|$

b) $f|_{|A \cup B|} = g|_{|A \cup B|}$

c) $f|_{|B \cup C|}$ is in general position with respect to some subdivision of $B \cup C$, and

d) $\forall\, L \leq K$, $g|_{|L|}$ is injective

$\Rightarrow f|_{|L|}$ is injective.

Heegaard splittings

Def A cpct, not necessarily connected $(n-1)$-mfld F is

<u>2-sided</u> in M M

$\exists$ an embedding $h: F \times [-1,1] \to M$

with $h(x,0) = x \quad \forall x \in F$

and $h(F \times [-1,1]) \cap \partial M = h(\partial F \times [-1,1])$

Lemma 2.1

If F is a compact $(n-1)$ mfld

properly embedded in an n-mfld

and if the image of

$$i_*: H_1(F;\mathbb{Z}_2) \to H_1(M;\mathbb{Z}_2)$$

is 0 then F is two sided in M.

Recall <u>Thm 1.8</u>: If a cpct polyhedron has two regular n'hoods (which are n-submanifolds) then they are homeomorphic.

Let us record two corollaries:

Corollary 2.9

If (T,h) is a combinatorial triangulation of a PL n-manifold M, and $L \leq T$ s.t. L is a vertex, then every regular n'hood of $h(|L|)$ in M is a PL n-cell.

Corollary 2.10

If M is a PL n-mfld, then every regular n'hood of ∂M in M is PL-homeomorphic to
$\partial M \times [0,1]$.

Proof of Lemma 2.2 (for $n=3$)

Take C, a connected component of F.

Since C is a submanifold of dimension 2, there is a triangulation T (combinatorial) in the PL structure of M

s.t. $\forall$ vertex v in C there is a subcomplex B of T containing v in its interior, which is a PL 3-cell, and

$|B| \cap F$ is the image of a 2-cell in B.

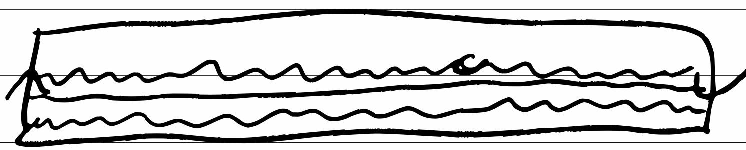
Passing to the second barycentric subdivision of T, we may assume that the star of $|B| \cap F$ is a regular n'hood of $|B| \cap F$.

Clearly, the star $\cong (|B| \cap F) \times [-1,1]$.

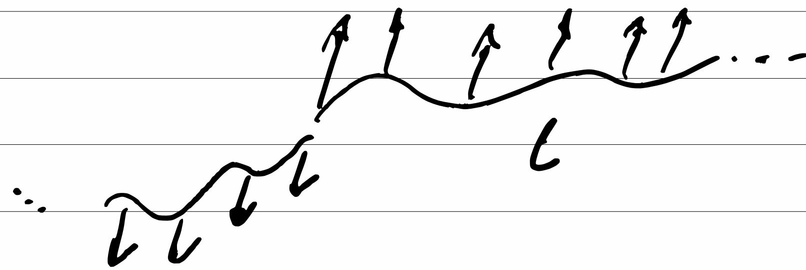
The union of these stars is a regular n'hood of C.

It might not however be homeomorphic to $C \times [-1,1]$:

example: Möbius band



The problem is that there is a loop l in C, s.t. moving the edge going in the positive direction along the loop changes it into a vector going in the negative direction.



Hence, there exists a loop ℓ' disjoint from F except at one point, where the intersection is transverse, which projects onto ℓ.

So ℓ' gives a cycle in $H_1(M; \mathbb{Z}/2)$ in the image of $H_1(F; \mathbb{Z}/2)$, which is 0 by assumption.

But the number of transverse intersections mod 2 is a homological invariant #

Def With notation as above, if F is two-sided and we have an embedding $F \times [-1,1] \xhookrightarrow{h} M$, the n-manifold $R = M \setminus h(F \times (-1,1))$ is <u>the result of cutting M along F.</u>

Since regular n'hoods are unique

up to homeomorphism, R is well-
-defined.

Moreover, using regular n'hoods of $F \times \{-1\}$ and $F \times \{1\}$, we see that $\exists$ a map $g: R \twoheadrightarrow M$ taking $R \setminus F \times \{-1, 1\}$ onto $M \setminus F$, and $g(h(x,-1)) = g(h(x,1)) = x \quad \forall x \in F$.

Cubes with handles.

Def Let M be a 3-manifold containing a collection $\{D_1, \ldots, D_n\}$ of pairwise disjoint, properly embedded 2-cells (which are known to be two-sided by Lemma 2.1) such that cutting M along $\cup D_i$ results in a 3-cell is called a cube with n handles.

By van Kampen's thm, $\pi_1(M) \cong F_n$.

Theorem 2.2

Suppose M_i is a cube with n_i handles, where $i \in \{1, 2\}$. Then M_1 is homeomorphic to M_2 iff $n_1 = n_2$ and either both are orientable or both are non-orientable.